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An Introduction to Wavelet Theory and Analysis

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An Introduction to Wavelet Theory and Analysis

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Abstract

This report reviews the history, theory and mathematics of wavelet analysis. Examination of the Fourier Transform and Short-time Fourier Transform methods provides information about the evolution of the wavelet analysis technique. This overview is intended to provide readers with a basic understanding of wavelet analysis, define common wavelet terminology and describe wavelet analysis algorithms. The most common algorithms for performing efficient, discrete wavelet transforms for signal analysis and inverse discrete wavelet transforms for signal reconstruction are presented. This report is intended to be approachable by non-mathematicians, although a basic understanding of engineering mathematics is necessary.

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An Introduction to Wavelet Theory and Analysis

1 Introduction

The theory behind wavelets has been developed during the last twenty to thirty years independently by mathematicians, scientists and engineers working in the areas of harmonic analysis theory (Calderon, 1964), filter bank theory (Esteban & Galand, 1977; Smith & Barnwell, 1986; Vetterli, 1984), and quantum mechanics (Aslaksen & Klauder, 1968). Morlet (1983) proposed the use of wavelets for analysis of seismic data and first coined the term "wavelets." From 1987 to 1992, synthesis of these cross-disciplinary approaches evolved into *wavelet analysis*. Wavelet analysis has been used in a variety of applications, including image compression (DeVore, Jawerth & Lucier, 1992), signal de-noising (Donoho & Johnstone, 1994), noise reduction (Esteban & Galand, 1977) speech and music processing (Kronland-Martinet, 1988), sound pattern analysis (Kronland-Martinet, Morlet & Grossmann, 1987) and sound synthesis (Miner, 1998).

Wavelets provide a tool for time-scale analysis of stationary (linear-time invariant) or nonstationary signals. Wavelets may be a more appropriate technique for analysis of real-world signals because the method captures the time-varying nature of these signals very effectively. For example, real-world sound signals describe the spatial and temporal course of an ecological event. As such, the sinusoidal components of a sound are not eternal in time, but rather they have a beginning, an end and, most likely, variations in time for the sound duration. Wavelets are finite in duration and therefore provide analysis of local signal features. Wavelet transforms maintain all the signal frequency and timing information. For these reasons, wavelet-based methods for processing non-stationary, real-world signals may provide better results than more traditional methods.

This report presents information on wavelet theory and the wavelet transform technique. The intention is to provide the basic concepts necessary for applying wavelets to problems. The content for this report was distilled from several books on wavelets (Daubechies, 1992; Ogden, 1997; Cohen & Ryan, 1995; Misiti, et al., 1996; Kaiser 1994; and Strang & Nguyen, 1996). These books provide a detailed mathematical treatment of the wavelet theory that is summarized here.

To understand wavelet analysis, it is helpful to relate the technique to the more traditional methods of Fourier Analysis and Short Time Fourier Analysis. Thus, section 2 provides a brief review of the mathematical foundations of the Fourier Transform (FT) method used in Fourier Analysis. The FT method is most useful when considering stationary signals. Most real-world signals are not stationary in time. A FT variation that captures some time-varying information by analyzing signal "windows" is the Short-Time Fourier Transform (STFT). Section 3 describes the STFT technique. The report culminates in Section 4 with a description of wavelet analysis, including the historical development and the wavelet transform mathematics. Section 5 contains the report summary and Section 6 contains references.

2 Fourier Theory

Jean Fourier developed the Fourier theorem in 1822 (Resnick & Halliday, 1960). The Fourier theorem is the basis of many signal analysis techniques including the Fourier Transform, the Short-Time Fourier Transforms (STFTs) and, more recently, wavelet analysis. The Fourier theorem states that all signals are made up of a combination of sine waves of varying frequency, amplitude and phase that may or may not change with time. The Fourier transform is a mathematical technique based on this theorem that breaks a signal up into its constituent sinusoidal components. This section briefly reviews the mathematics of the Fourier Transform method.

2.1 The Fourier Transform

The Fourier Transform (FT) is a method for decomposing a time domain signal into its constituent frequency components. The standard continuous FT pair of equations is as follows (Oppenheim & Schafer, 1989).

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt \tag{2.1}$$

$$f(t) = \int_{-\infty}^{+\infty} F(\omega) e^{j\omega t} d\omega \tag{2.2}$$

where t is time, ω is radian frequency, $f(t)$ is the input signal, and $F(\omega)$ is the Fourier spectrum evaluated on the sine and cosine basis functions as indicated by $e^{j\omega}$. The relation between radian frequency, ω, and frequency in Hertz, f, is:

$$\omega = 2\pi f \tag{2.3}$$

All time domain signals no matter how complex, can be broken into constituent sinusoidal components using Equation 2.1. The result is a set of values, $F(\omega)$ (known as the Fourier spectrum of the signal) indicating the frequency components and respective amplitudes. The FT results do not provide explicit information about time duration, onset or offset of frequency components, because the information is spread across the Fourier spectrum. This is the case because the sine and cosine basis functions of the FT have *infinite support* (non-zero across an infinite interval). In this sense, the FT provides a global picture of the signal frequency content and timing. Thus, the transform is most applicable for *stationary* signals that do not vary over time. Signals decomposed using Equation 2.1 can be reconstructed through the Inverse Fourier Transform (IFT) shown in Equation 2.2. The result of the IFT is a time domain signal, $f(t)$, identical in content to the original signal.

2.2 The Discrete Fourier Transform

Discrete-time signals are those in which the independent and/or dependent variable has discrete values. Discrete-time signals are represented as sequences of numbers. Typically, real-world signals, such as sound, are digitized prior to analysis; thus, the time and amplitude are discrete. The equations for computing the Discrete Fourier Transform (DFT) coefficients, $X(k)$, and the Inverse Discrete Fourier Transform (IDFT) for an input signal $x[n]$ are as follows (Oppenheim & Schafer, 1989):

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} \tag{2.4}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\omega})e^{j\omega n}\,d\omega \tag{2.5}$$

where n is the discrete time index in samples and ω is the discrete radian frequency (ranging from $-\pi$ to $+\pi$). The sampling frequency, f_s, is related to ω by $\omega = \frac{2\pi f}{f_s}$.

In practice, real-world signals are nonstationary, meaning that the signal properties (amplitudes, frequencies and phases) change with time. The discrete FT does not represent the time-varying signal information in a manner that provides easy access to individual signal components. To overcome this shortcoming, the Short-Time FT (STFT) has been developed and is useful for real-world signal analysis. The next section describes the STFT.

3 The Short-Time Fourier Transform

A variation of the FT is the *windowed* FT, *time-dependent* FT or the *Short-Time Fourier Transform* (STFT) developed by Denis Gabor (Cohen & Ryan, 1995). The STFT views (and analyzes) the input signal in sections through a moving window function. This technique provides analysis of signals with time-varying information; however, the analysis resolution is limited by the choice of window size. This section describes the STFT technique and provides an example use of the STFT in music synthesis.

3.1 Mathematics of the Short-time Fourier Transform

The STFT method calculates a *frame* of Fourier coefficients by applying the FT to a localized time-slice of the signal as seen through a window function. The window is moved or "hopped" over by a specified amount of time, and then another windowed FT is performed. The discrete STFT is represented mathematically as follows (Oppenheim & Schafer, 1989; Serra & Smith, 1990):

$$X(l,\omega) = \sum_{n=-\infty}^{\infty} x[n+lH]w[n]e^{-j\omega n} \qquad l = 0,1,2 \tag{3.1}$$

where $x[n]$ is a discrete input signal, l is a particular frame, H is the hop size, $w[n]$ is the window function, ω is the radian frequency and $X(l, \omega)$ are the resulting set of FT coefficients for the frame.

In Equation 3.1, the discrete analysis window function, $w[n]$, determines the portion of the input signal to be processed at a particular frame l. H is a constant that represents the hop-size, or amount to move the signal. The STFT computes a FT spectrum on a window-sized portion of the input signal, $x[n]$, at every frame l. The signal is advanced by the hop-size, H, thereby "sliding" it past the window function. Thus, at each frame, l, a different portion of the signal is viewed through the window and is analyzed with a FT.

The choice of analysis window determines the smoothness of the spectrum and the detectability of different sinusoidal components. The most commonly used windows are the Rectangular, Hamming, Hanning, Kaiser, Blackman and Blackman-Harris. Harris (1978) provides a description of these window functions and the trade-offs for each. The characteristics of the analysis window remain fixed for the duration of the STFT analysis.

3.2 Inverse Short-time Fourier Transform Methods

The inverse STFT is the process of reconstructing the original signal from an STFT decomposition. Allen & Rabiner (1977) provide an in-depth description of two inverse STFT methods, the filter-bank summation (FBS) method and the overlap-add (OLA) method. The methods are complementary and have a formal duality.

The FBS approach modulates the $X(n, \omega)$ STFT coefficients back to frequency ω, and then sums the results over all frequencies. This is reasonable because $X(n, \omega)$ is a low-pass, band-centered representation of the input signal for any frequency, ω.

The OLA approach is based on the traditional FT interpretation of the STFT. This method is often superior to FBS when modifications are made to the STFT results prior to synthesis. The OLA method is based on the fact that the $X(l, \omega)$ STFT coefficients are essentially FTs of the sequence

$$\hat{x}_l[n] = x[n + lH]w[n] \tag{3.2}$$

where $\hat{x}_l[n]$ represents the FT of the windowed signal at frame l. Thus, the original signal samples can be recovered by taking the inverse FT of $X(l, \omega)$ and dividing out the window function (provided the window has at least one nonzero sample). The normalized inverse STFT follows directly from the FT synthesis equation as follows:

$$x_l[n] = \frac{1}{2\pi w[n]} \int_{-\pi}^{+\pi} X(l,\omega) e^{j\omega n} d\omega, \quad -\infty < n < \infty \tag{3.3}$$

Equation 3.3 shows that an inverse FT is performed on each analysis frame result. The complete signal is reconstructed by summing the l frames of inverse FT results:

$$x[n] = \sum_{l} x_l[n] \tag{3.4}$$

3.3 Discussion

The STFT examines an input signal in time segments as determined by the analysis window length. For nonstationary signals, this is an improvement over the traditional FT, because some of the time-based information is maintained explicitly according to frames. However, the standard STFT technique assumes that the input signal is stationary within the duration of the analysis window. For real-world signals, this is often not the case. Serra & Smith (1990) propose a method for tracking the time-varying information between successive windowed results obtained from an STFT analysis. The method assumes that salient features are identified by peak frequency responses when a window of the signal is analyzed. They employ a *peak detection* algorithm for isolating the salient frequencies and a *peak continuation* algorithm for tracking the salient signal features between window frames. The signal information that is not tracked is modeled as stochastic noise. To model the noise, a *residual,* or error between the salient signal features and the original input signal, is calculated. Each analysis frame yields a different residual. An envelope approximation algorithm estimates the overall shape of the residual noise for each frame. During reconstruction, the residual envelopes are used as filters on a pseudo-random noise source. These noise sources are combined with the inverse STFT of the salient signal features to produce efficient and realistic signal synthesis.

Serra and Smith's approach works well for modeling musical instruments, because the basic assumption, that frequency peaks indicate salient features, is often true for musical tones. However, this is not necessarily the case for all real-world signals. For example, the stochastic noise components strongly characterize real-world environmental sound signals. Thus, the salient signal features of environmental sounds are not illuminated by a STFT analysis and cannot be tracked from frame to frame using the peak detection scheme. Furthermore, Serra and Smith's envelope approximation method for modeling the stochastic signal components is not appropriate for modeling the stochastic characteristics of real-world environmental sounds. The method is adequate for capturing supplementary information for a primarily pitched sound and adds a sense of realism to Serra and Smith's musical instrument models; however, the method does not have the time-resolution necessary for capturing the perceptual essence of real-world stochastic-based signals. The next section describes the mathematical theory behind wavelets and discusses how wavelets are often more appropriate for modeling the time-varying nature of real-world signals.

4 Wavelet Theory

This section presents an overview of wavelet theory and wavelet analysis. Standard notations are defined in the first section, followed by the historical development of wavelets. Next the mathematics of the discrete wavelet transform are presented including an illustrative example of how to perform a wavelet analysis. Next, the relationship between wavelets and scaling functions is defined. Understanding this relationship is important for understanding the basis of the inverse wavelet transform that provides perfect signal reconstruction. The final sections describe efficient algorithms for performing discrete wavelet transforms and inverse discrete wavelet transforms.

4.1 Definitions and Notation

Th following definitions and notations are used throughout this section:

$\Re$ the set of real numbers $(-\infty, \infty)$.

Z the set of integers $\{\ldots,\ -2,\ -1,\ 0,\ 1,\ 2,\ \ldots\}$.

L^2 the Hilbert function space, or the set of functions that are square - integrable, $L^2(I)$

$L^2(I)$ square - integrable functions $\{f : \int_I f^2(x)dx < \infty\}$; set of signals with finite energy.

$l^2(Z)$ is square - summable: $\{m_k\}$ is an l^2 sequence if $\sum_{k \in Z} m_k^2 < \infty$.

$\langle f, g \rangle$ the L^2 inner product of two functions: $\langle f, g \rangle = \int f(x)g(x)dx$.

$\|f\|$ the L^2 norm: $\|f\|^2 = \langle f, f \rangle$.

Compact support: a function's impulse response, $\{h_n\}, n \in Z$ is finite.

Orthogonal functions: Two functions $f_1, f_2 \in L^2$ are orthogonal if $\langle f_1, f_2 \rangle = 0$.

Orthonormal functions: A function sequence $\{f_i\}$ is orthonormal if:

1) the f_i's are pairwise orthogonal
2) the L^2 norm, $\|f_i\|$, equals 1 for all i.

Haar Wavelet Function: $$\psi(x) = \begin{cases} 1, & 0 \le x < 1/2 \\ -1, & 1/2 \le x < 1 \\ 0, & \text{otherwise} \end{cases}$$

FIR (Finite Impulse Response) filters: the impulse response is finite in length (i.e. is zero outside a finite interval).

4.2 Historical Development

Wavelet analysis is the next logical step for analysis of time-varying, real-world signals. Wavelet analysis is a windowing technique similar to the STFT but with variable-sized windows. As with the FT and STFT methods, wavelet analysis consists of signal decomposition (wavelet transform) and reconstruction (inverse wavelet transform) phases. This section describes the historical evolution of wavelet analysis and describes the commonly used terminology.

Alfred Haar (1910) is credited with the first use of a wavelet, although the term *wavelet* was not coined until Morlet used it in his signal processing work of 1983. The wave shape used by Haar is now known as the *Haar wavelet*. It is the simplest wavelet possible and resembles a step function. The Haar wavelet is discontinuous and therefore consequently yields poor frequency localization. Haar wavelets are helpful for developing a basic understanding of wavelet analysis but are not often used in practice.

The Haar wavelet is an example of an *orthonormal* wavelet. Orthonormal wavelets have specific properties that provide a means for efficient decomposition of a signal. Orthonormal wavelet functions define a specific set of filters for efficient signal decomposition and reconstruction. With the high speed of today's computer systems, real-time wavelet synthesis is feasible using these wavelet functions. Stromberg (1982) is often credited with the development of *orthonormal* wavelets; however, the orthonormal wavelet system introduced by Yves Meyer in 1985 received more recognition and became popularly known as the *Meyer basis* (Meyer, 1993).

Esteban & Galand (1977) in their subband coding research, proposed a filtering scheme that did not introduce signal aliasing. With this scheme, signals are filtered into low and high frequency components with a pair of filters. The filters are mirror images with respect to the middle, or quadrature frequency, $\pi/2$ (Strang & Nguyen, 1996). Filters chosen according to this scheme are called *quadrature mirror filters* (QMFs) or *conjugate quadrature filters* (CQFs). Mathematical derivation of QMFs will be described in Section 4.4. Orthonormal wavelet bases developed with QMFs provide exact signal reconstruction.

Ingrid Daubechies (1988) constructed wavelet bases with *compact support* meaning that the wavelets are non-zero on an interval of finite length (as opposed to the infinite interval length of the FT's sine and cosine bases functions). Compactly supported wavelet families accomplish signal decomposition and reconstruction using only Finite Impulse Response (FIR) filters. This development made the discrete-time wavelet transform a reality.

Stephane Mallat (1989) proposed the Fast Wavelet Transform (FWT) algorithm for the computation of wavelets in 1987. This technique was similar to noise reduction techniques developed in the 1970s (Esteban & Galand, 1977). Both of these techniques are unified by the concept of *multiresolution analysis*. In fact, the concepts of wavelet analysis are tightly coupled with those of multiresolution analysis. Multiresolution analysis (MRA) is based on the concept that objects can be examined using varying levels of resolution. An analogy is the multiple-level-of-detail concept in computer graphics. When a viewer is far away from a graphical object, a low level of detail can be used to render the object. As the viewer nears the object, a higher level of detail is required so that the object appears realistic. This *zoom-in, zoom-out* property of MRA serves as one of the basic properties for wavelet analysis. Orthonormal wavelet functions are derived from *scaling functions* (described in Section 4.4) that satisfy the properties of MRA. Briefly, a multiresolution analysis 1) is an increasing sequence of closed, nested subspaces $\{V_j\}_{j \in Z}$ that tends to $L^2(\Re)$ as j increases, 2) satisfies a *twin-scale* relation (defined in Section 4.4) linking successive decomposition levels and 3) contains an orthonormal basis derived from a function, ϕ, and its integer translates $\{\phi(x-b)_{b \in Z}\}$ (Cohen & Ryan, 1995).

The next sections present a summary of the wavelet transform, wavelet function, scaling function and coefficient calculations. The discussion in these sections pertains to orthonormal wavelets in the MRA framework and compactly supported scaling functions. Discussion of wavelets not satisfying these criteria is beyond the scope of this report.

4.3 The Discrete Wavelet Transform

Similar to the STFT approach, wavelets analyze the input signal in sections by translation of an *analysis function.* With STFT, the analysis function is a window. The window is translated in time but is not otherwise modified. The wavelet approach replaces the STFT window with a wavelet function, ψ. The wavelet function is *scaled* (or *expanded* or *dilated*) in addition to being translated in time. The ψ is often called a *mother wavelet* because it "gives birth" to a family of wavelets through the dilations and translations. A generalized wavelet family, $\psi_{a,b}$, described in the normalized form is:

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right) \tag{4.1}$$

where a represents the scale and b represents the translation parameters.

The scale parameter, a, indicates the *level* of analysis. Small values of a provide a local, fine grain or high frequency, analysis while large values correspond to large scale, coarse grain or low-frequency, analysis. Changing the b parameter moves the time localization center of each wavelet; each $\psi_{a,b}(x)$ is localized around $x = b$.

Typically, the scale factor between levels increases by two. Thus, scaling is also known as *dilation.* Widely used a and b parameter settings that create an orthonormal bases are $a = 2^j$ and $b = 2^j k$ $(j,k \in Z)$. The wavelet family then becomes:

$$\psi_{j,k}(x) = 2^{-j/2}\psi(2^{-j}x-k) \tag{4.2}$$

The wavelet is not necessarily symmetric, but in order for perfect reconstruction to be possible, it does satisfy $\int \psi(x)dx = 0$. Other properties of the wavelets discussed in this report are orthonormality and compact support. Wavelet families that satisfy these conditions are the Daubechies wavelets (often denoted by *dbN*, where *N* is the wavelet order), Symlet wavelets (*symN*) and Coiflet wavelets (*coifN*). Other wavelet families exist, including the symmetrical types of the Morlet wavelet, Meyer wavelet and Mexican Hat wavelet. Construction of new wavelet families is an active research area. The most commonly used wavelets are the Daubechies family, although other wavelet families may prove more advantageous for decomposition of real-world signals. The choice of wavelet type is highly application specific.

Analogous to the STFT, the wavelet transform calculates wavelet coefficients by taking the inner product of an input signal, $f(x)$, with a function, that is in this case the wavelet family, $\psi_{j,k}(x)$. The continuous time, discrete wavelet transform (DWT) is:

$$D_{j,k} = \left\langle f, \psi_{j,k} \right\rangle = 2^{-j/2} \int_{-\infty}^{+\infty} f(x)\psi\left(2^{-j}x - k\right)dx \tag{4.3}$$

where $D_{j,k}$ are the *wavelet coefficients.* In the wavelet vernacular, the wavelet coefficients are called *details*. The next section makes the significance of this clear by examining how the wavelet functions are obtained. From an intuitive perspective, the wavelet coefficients are measures of the goodness of fit between the signal and the wavelet. Large coefficients indicate a good fit. Figure 4.1 graphically illustrates the wavelet transform steps on an arbitrary signal using the Daubechies 4 (db4) wavelet type.

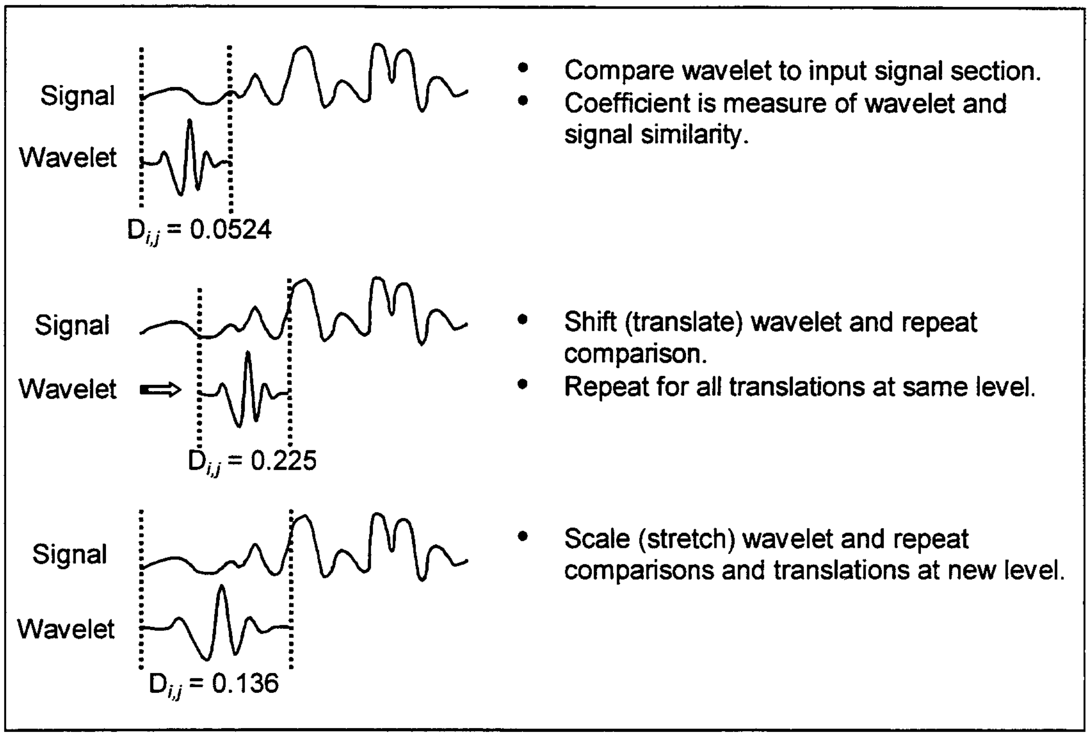

Figure 4.1 Illustration of wavelet transform steps to calculate wavelet coefficients, D_{ij} (example uses Daubechies 4 (db4) wavelet type).

4.4 Wavelets and the Scaling Function

As described earlier, changing the mother wavelet's dilation, j, and translation, k, parameters, creates a family of wavelets, $\psi_{j,k}$. Filtering the input signal by $\psi_{j,k}$ results in a set of detail coefficients that represent the high-frequency signal information. This section describes how to obtain the mother wavelet using *a scaling function.*

Wavelet functions are constructed from a *father wavelet*, or scaling function, ϕ. The scaling function is not any arbitrary shape, but rather it is a finite duration function with integral $\int \phi(x)dx = 1$, satisfying the properties of a multiresolution analysis (MRA), as stated earlier. From the scaling function, ϕ, it is possible to construct an orthonormal wavelet, ψ, such that a signal can be decomposed (analyzed) and reconstructed exactly and efficiently. The development of this relationship is briefly summarized here.

There exists a *twin-scale relation* (also known as the *dilation* or *refinement equation*) that relates MRA functions at successive levels:

$$\phi(x) = \sqrt{2} \sum_{k \in Z} h_k \phi(2x - k) \tag{4.4}$$

where $\phi(x)$ is the scaling function and h_k is a square-summable sequence whose elements are obtained from the inner product of two levels of scaling functions:

$$h_k = \left\langle \phi_{j+1,0}, \phi_{j,k} \right\rangle \tag{4.5}$$

The sequence $\{h_k\}$ represents the coefficients of the scaling function filter. If a scaling function is selected from one of the known family of wavelets, the scaling filter coefficients are known. The scaling filter is a low-pass, FIR filter. The filter has the properties of $\sum h_k = 1$ and normalization of $\sqrt{\sum h_k^{\,2}} = \frac{1}{\sqrt{2}}$.

Using the twin-scale relation and the MRA properties, a general equation for calculating the scaling function at any level $j+1$ given level j is given by the equation:

$$\phi_{j+1,0}(x) = \sum_{k \in Z} h_k \phi_{j,k}(x) \tag{4.6}$$

where $\phi_{j,k}(x)$ is the scaling function at level j with translation index k, and $\phi_{j+1,0}$ is the next lower level scaling function (larger indices, j, correspond to expansion of ϕ).

From the scaling function, the wavelet function, ψ, is calculated as follows:

$$\psi(x) = \sqrt{2} \sum_{k \in Z} g_k \phi(2x - k) \tag{4.7}$$

where $\psi(x)$ represents the mother wavelet (top most wavelet) and g_k represents the *wavelet filter* coefficients defined by:

$$g_k = (-1)^k h_{1-k} \tag{4.8}$$

Thus, the wavelet function is obtained by convolving the scaling function with the reversed and alternating signed form of the scaling filter. The wavelet calculated by Equation 4.7 is orthogonal to the scaling function. The general equation for calculating the wavelet function at any level j is given by:

$$\psi_{j+1,0}(x) = \sum_{k \in Z} g_k \phi_{j,k}(x) \tag{4.9}$$

Figure 4.2 shows an example of a wavelet and corresponding scaling function, specifically the example shows the Daubechies 4 (db4) functions. As the figure shows, the wavelet function has high-frequency oscillations and the scaling function is lower in frequency. Thus, the wavelet function creates a high-pass wavelet filter (g_k) that provides the *detail coefficients*. The scaling function creates a low-pass filter (h_k) that provides the *approximation coefficients*. The terms are appropriate; consider a speech signal for example. A speech segment can be understood when high-frequency details are suppressed; however, if the low-frequency signal approximations are removed, a human listener cannot interpret the speech signal.

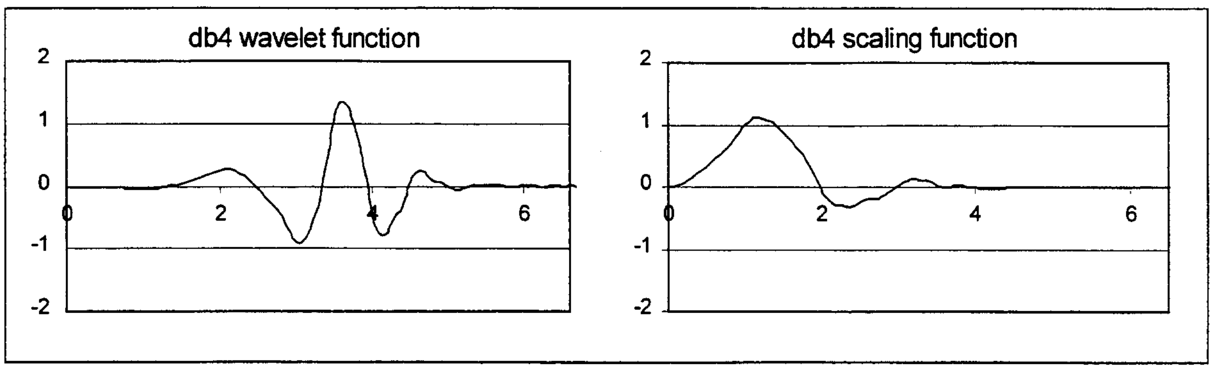

Figure 4.2 Daubechies 4 (db4) wavelet and scaling functions.

Deriving the wavelet filter coefficients by Equation 4.8 forces the wavelet and scaling filters to be quadrature mirror filters (QMF) of each other and makes perfect signal reconstruction possible. The wavelet filter is the mirror reflection of the scaling filter with alternating signs. For example, if the scaling filter coefficients are $h_k = \{a,b,c,d\}$, then the wavelet filter coefficients are $g_k = \{d,-c,b,-a\}$. Because the QMF property is critical, often wavelet researchers start with the design of the QMF filters and let the filter coefficients determine the shape of the wavelet and scaling functions. Misiti et al. (1996) describes the algorithm for obtaining the ϕ and ψ function shapes from the respective filters as repeatedly convolving an upsampled wavelet with the scaling filter.

How are the approximation coefficients calculated? Recall from Equation 4.3 that the detail coefficients are created by convolving the input signal with the wavelet function. The approximations coefficients are calculated in the same way, by taking the inner product of the input signal, f, and the family of dilated, j, and translated, k, scaling functions:

$$A_{j,k} = \langle f, \phi_{j,k} \rangle = 2^{-j/2} \int_{-\infty}^{+\infty} f(x)\phi(2^{-j}x - k)dx \tag{4.10}$$

Equations 4.3 and 4.10 define the procedure for complete signal decomposition using wavelets. Although these equations can be implemented algorithmically and would provide accurate results, they do not provide efficient signal decomposition. The next section describes an efficient decomposition procedure that uses convolution rather than integration for calculating the detail and approximation coefficients.

4.5 Efficient Wavelet Decomposition Algorithm

As described in the previous sections, a signal is decomposed using the wavelet transform technique into two sets of coefficients called approximations ($A_{j,k}$) and details ($D_{j,k}$). The approximation coefficients represent the low frequency and the detail coefficients represent the high-frequency signal components. Calculating the detail and approximation coefficients through integration as shown in Equations 4.3 and 4.10 is time consuming, especially as the decomposition algorithm is applied repeatedly to intermediate sets of coefficients (known as *multi-level decomposition*). A more efficient algorithm results from calculating the $A_{j,k}$ and $D_{j,k}$ coefficients through convolution of the input signal with the scaling filter $\{h_k\}$ and wavelet filter $\{g_k\}$ respectively. This recursive decomposition algorithm is sometimes referred to as the *cascade algorithm* (Daubechies, 1992) or the *pyramid algorithm* (Mallat, 1989). It is key to the fast wavelet transform algorithm.

Using the relations in Equations 4.10 and the Dilation Equation in 4.6, an efficient decomposition algorithm for computing the $A_{j,k}$ coefficients is obtained (Ogden, 1977):

$$A_{j+1,k} = \sum_n h_{n-2k} A_{j,n} \tag{4.11}$$

where j is again the level (or scale), k is the translation index, and h_k is the scaling function filter coefficients as in Equation 4.5. This equation says that lower-level approximation coefficients ($A_{j+1,k}$) are computed recursively given the approximation coefficients at a higher level (A_j).

Similarly, a twin-scale relationship for computing the $D_{j,k}$ coefficients is obtained using the relation in Equation 4.9 resulting in the decomposition formula:

$$D_{j+1,k} = \sum_n g_{n-2k} A_{j,n} \tag{4.12}$$

where g_k is the wavelet filter coefficients as defined in Equation 4.8. This equation says that lower-level detail coefficients ($D_{j+1,k}$) are computed from higher-level approximation coefficients, (A_j). Recursive application of these decomposition formulas provides a means for obtaining lower level detail and approximation coefficients once the highest level approximation coefficients are calculated. The input signal provides the top level (finest grain) approximation coefficients, (A_0). Figure 4.3 shows a pictorial of the decomposition process.

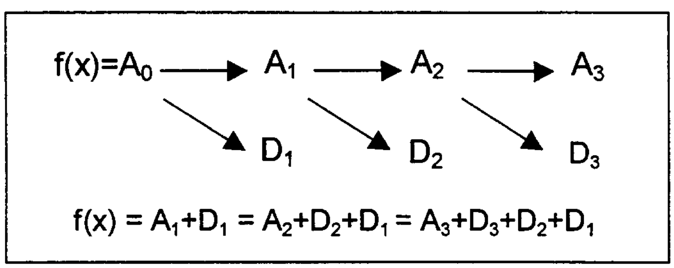

Figure 4.3 Schematic of wavelet decomposition algorithm. Lower level Approximation (A_j) and Detail (D_j) coefficients are obtained from the highest level A_0 coefficients. A_0 is the input signal f(x).

Notice in Equations 4.11 and 4.12, changing the translation index, k, by 1 results in the indices of the $\{h_k\}$ and $\{g_k\}$ sequences being offset by two. Thus, there are half as many coefficients at level $j+1$ as there are at level j. The result is a *downsampling* of the coefficient vectors by a factor of two in the decomposition algorithm. The downsampling and recursive nature of the algorithm are important components of the fast wavelet transform algorithm.

4.6 *Inverse Discrete Wavelet Transform*

The wavelet decomposition algorithm is reversible and provides exact signal reconstruction. The inverse discrete wavelet transform (IDWT) provides signal reconstruction or synthesis. Lower level approximation and detail coefficients combine to create higher level coefficients. Figure 4.4 shows the reconstruction process.

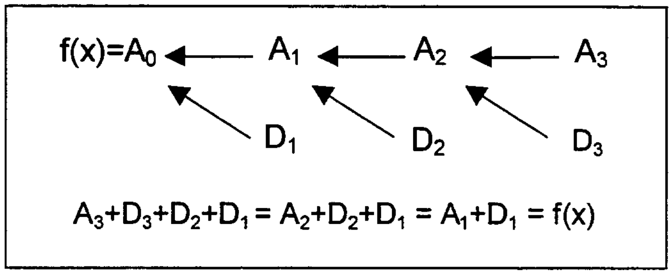

Figure 4.4 Schematic of wavelet reconstruction algorithm. Lower level Approximation (A_j) and Detail (D_j) coefficients combine to reconstruct signal.

The discrete wavelet reconstruction formula using the wavelet filter $\{g_k\}$ and scaling filter $\{h_k\}$ is as follows:

$$A_{j,k} = \sum_m h_{k-2m} A_{j+1,m} + g_{k-2m} D_{j+1,m} \tag{4.13}$$

Thus, the approximation coefficients ($A_{j,k}$) at any level can be computed from one set of low-level scaling function coefficients ($A_{j+1,m}$) and all the intermediate wavelet coefficients ($D_{j+1,m}$). In order to provide perfect reconstruction, the h and g reconstruction filters are mirror images of the decomposition filters. For example, if the decomposition filter H = h_k = {a,b,c,d}, then the reconstruction filter H' = h_k' = {d,c,b,a}. Consequently, the pair of reconstruction filters are also QMFs. Notice that the subscript on the h and g filters is $k\text{-}2m$. The effect of the $2m$ subscript is that the coefficient vectors A and D are upsampled (zeros inserted at every other location) prior to convolution with the filters. This is analogous to the downsampling operation in the decomposition process. The upsampled and filtered coefficient vectors are then added together to create the next higher level $A_{j,k}$ vector. This process is repeated recursively to recreate the original input signal.

4.7 Discrete Wavelet Transform Summary

Overall, the DWT decomposes a signal into high (A) and low (D) frequency coefficients through filtering and downsampling operations. Recursively iterating the decomposition steps breaks the signal into lower level (coarser grain) coefficient sets.

Manipulating wavelet coefficients prior to signal reconstruction changes the original signal. The original signal can be modified, enhanced or de-noised through various coefficient manipulation operations. In Miner (1998), subtle and compelling sound variations were obtained by manipulating the coefficients obtained from wavelet decompositions. Thresholding wavelet coefficients is an effective method of de-noising or compressing a signal or an image (Misiti, et al., 1996).

Without coefficient manipulations, the original signal can be reconstructed exactly through the IDWT. A modified version of the original signal is obtained through the IDWT if coefficient manipulations have been performed. The reconstruction steps involve upsampling and filtering operations with filters that are mirror reflections of the decomposition filters. The filter pairs (decomposition and reconstruction) are quadrature mirror filters (QMF) enabling perfect reconstruction. Figure 4.5 shows a pictorial summary of a single level wavelet decomposition and reconstruction operation. This figure indicates the functions necessary for computer implementation of the wavelet transform, including filtering, downsampling and upsampling operations.

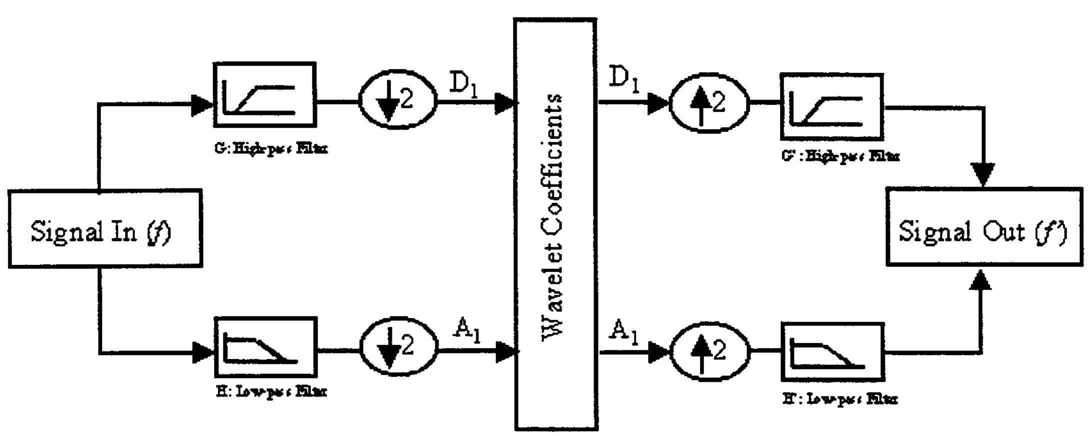

Figure 4.5 Wavelet decomposition and reconstruction process. Wavelet filter, G, produces high-frequency details (D). Scaling function filter, H, produces low-frequency approximations (A).

The wavelet filter construction is demonstrated through an example. Consider the Daubechies 3 (db3) scaling filter with six coefficients (row 1, Table 4.1). Coefficients of the low-pass reconstruction filter, H', are normalized versions of the scaling filter coefficients (H' = db3/(norm(db3)). Reversing the order of H' and changing the even coefficient signs creates the high-pass reconstruction filter, G'. Reversing the order of H' and G' yields the low-pass (H) and high-pass (G) decomposition filters. Figure 4.6 contains plots of the impulse responses for the db3 scaling function and filters.

Vector Name	**Coefficient Index**					
	1	**2**	**3**	**4**	**5**	**6**
db3	0.2352	0.5706	0.3252	-0.0955	-0.0604	0.0249
H	0.0352	-0.0854	-0.1350	0.4599	0.8069	0.3327
G	-0.3327	0.8069	-0.4599	-0.1350	0.0854	0.0352
H'	0.3327	0.8069	0.4599	-0.1350	-0.0854	0.0352
G'	0.0352	0.0854	-0.1350	-0.4599	0.8069	-0.3327

Table 4.1 Wavelet filter construction example showing Daubechies wavelet 3 (db3) scaling filter coefficients and corresponding low-pass (H) and high-pass (G) decomposition and reconstruction (H' and G' respectively) filters.

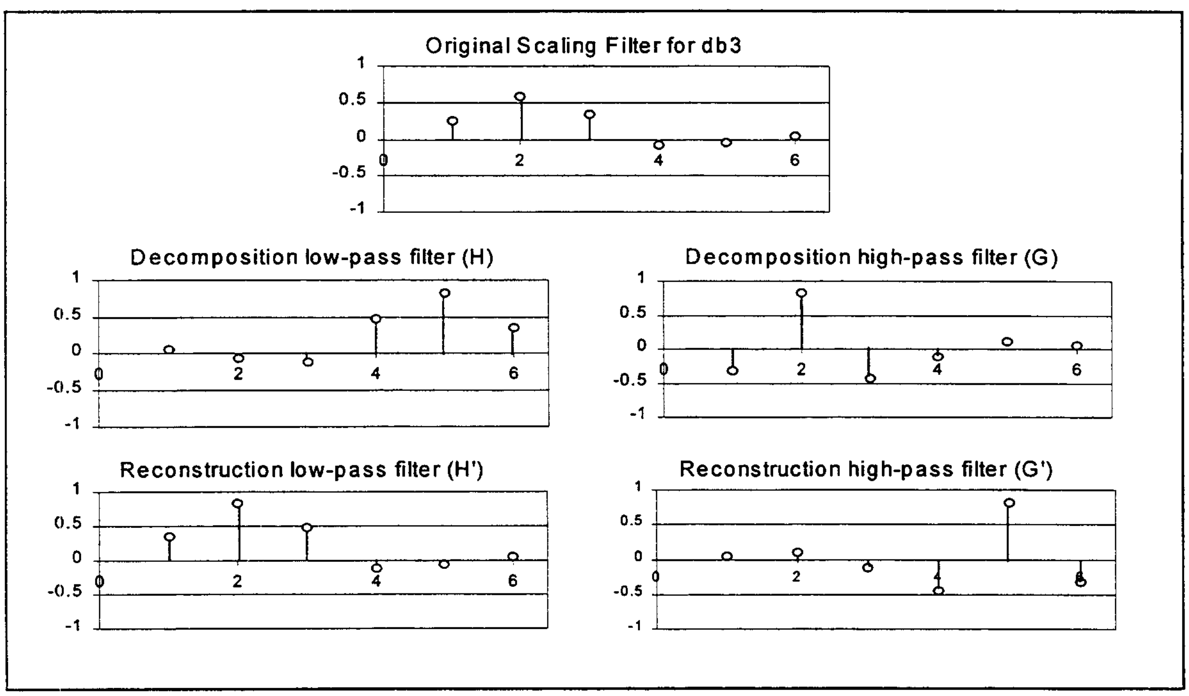

Figure 4.6 Plots of impulse responses for Daubechies wavelet db3 scaling function filter and corresponding low-pass and high-pass filters from Table 4.1

Wavelet analysis, including decomposition and reconstruction functions, are becoming readily available in a variety of software systems. Matlab™ has extensive support of these functions in their wavelet toolbox (Misiti, et al., 1996). Wavelet support is available for Intel's MMX processors in the signal processing libraries of the VTune plug-in for Visual C/C++ (VTune, 1997). In addition, a variety of freeware and shareware implementations of the discrete wavelet transform algorithms are available on the world wide web. Thus, software developers today need not write their own wavelet analysis routines from scratch.

5 Summary

The wavelet transform is often times advantageous for signal analysis and synthesis over the traditional Fourier and Short-time Fourier transform techniques. The major benefit to using wavelets is that variable length, finite filters are used for signal analysis, thus, local, time-varying information is captured explicitly. Discrete wavelet transform algorithms are well defined and provide efficient signal decomposition and perfect signal reconstruction. The wavelet decomposition coefficients, approximations and details, obtained from a wavelet analysis provide a means for tracking the signal characteristics over time and frequency. These coefficients can be manipulated to change the original signal characteristics if desired. The wavelet reconstruction algorithm recursively combines the coefficient groups to obtain the original signal, or a variation of the original signal (if coefficients are modified after decomposition).

Overall, this report has presented a review of the basic theory behind wavelet analysis. The evolution of wavelet analysis was presented from the perspective of the Fourier Transform and Short-time Fourier Transform techniques. The discrete wavelet transform decomposition and reconstruction algorithms were presented to provide developers with a basic understanding of the approach. The reference section includes sources that can provide the reader with more detailed information on wavelet analysis, theory and algorithms.

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